

# Analysis of a drift-diffusion model with velocity saturation for spin-polarized transport in semiconductors

Nicola Zamponi

*Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstraße 8–10,  
1040 Wien, Austria*

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## Abstract

A system of drift-diffusion equations with electric field under Dirichlet boundary conditions is analyzed. The system of strongly coupled parabolic equations for particle density and spin density vector describes the spin-polarized semi-classical electron transport in ferromagnetic semiconductors. The presence of a nonconstant and nonsmooth magnetization vector, solution of the Landau-Lifshitz equation, causes the diffusion matrix to be dependent from space and time and to have in general poor regularity properties, thus making the analysis challenging. To partially overcome the analytical difficulties the velocity saturation hypothesis is made, which results in a bounded drift velocity. The global-in-time existence and uniqueness of weak solutions is shown by means of a semi-discretization in time, which yields an elliptic semilinear problem, and a quadratic entropy inequality, which allow for the limit of vanishing time step size. The convergence of the weak solutions to the steady state, under some restrictions on the parameters and data, is shown. Finally the higher regularity of solutions for a smooth magnetization in two space dimensions is shown through a diagonalization argument, which allows to get rid of the cross diffusion terms in the fluid equations, and the iterative application of Gagliardo-Nirenberg inequalities and a generalized version of Aubin lemma.

**Keywords:** drift-diffusion equations, global existence, cross diffusion, spin transport, charge transport, velocity saturation

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## 1. Introduction

In this paper, we study global existence, uniqueness, long-time behavior and regularity of the solutions to the following drift-diffusion equations:<sup>1</sup>

$$\partial_t n_0 - \partial_i \left( \frac{D}{\eta^2} (J_0^i - p m_s J_s^i) \right) = 0 \quad \text{in } \Omega \times (0, T), \quad (1)$$

$$\begin{aligned} \partial_t n_k - \partial_i \left( \frac{D}{\eta^2} (-p m_k J_0^i + (\eta \delta_{ks} + (1 - \eta) m_k m_s) J_s^i) \right) \\ - 2\gamma \varepsilon_{ijk} n_i m_j + \frac{n_k}{\tau} = 0 \quad (k = 1, 2, 3) \quad \text{in } \Omega \times (0, T), \end{aligned} \quad (2)$$

$$J_0^i = \partial_i n_0 - v_i n_0, \quad J_s^i = \partial_i n_s - v_i n_s, \quad (1 \leq i, s \leq 3) \quad \text{in } \Omega \times (0, T), \quad (3)$$

where  $\partial_t = \partial/\partial t$ ,  $\partial_i = \partial/\partial x_i$  are the partial derivatives with respect to  $t$  and  $x_i$ , for  $i = 1, 2, 3$ , respectively,  $\Omega \subset \mathbb{R}^d$  with  $d \leq 3$  is a bounded domain,  $D(x)$  is the diffusion coefficient,  $p(x) \in (-1, 1)$  is the spin polarization,  $\eta(x) = \sqrt{1 - p(x)^2}$ ,  $\vec{v}(x, t) = (v_1(x, t), v_2(x, t), v_3(x, t))$  is the scaled drift velocity,  $\vec{m}(x, t) = (m_1(x, t), m_2(x, t), m_3(x, t))$  is the magnetization vector and  $\varepsilon_{ijk}$  is the Levi-Civita tensor. The system describes the evolution of a spin-polarized electron charge

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<sup>1</sup>We adopt the Einstein convention of sum over repeated indexes.

distribution with charge density  $n_0(x, t)$ , spin density vector  $\vec{n}(x, t) = (n_1(x, t), n_2(x, t), n_3(x, t))$  in a ferromagnetic semiconductor. The drift velocity is related to the electric field  $-\nabla V(x, t)$  by the following relation [16, 19]:

$$v_i = -\mu(|\nabla V|)\partial_i V \quad (i = 1, 2, 3), \quad (4)$$

where the function  $\mu : [0, \infty) \rightarrow (0, \infty)$  is the so-called *electron mobility*, and the electric potential  $V(x, t)$  is self-consistently given by the Poisson equation [14]:

$$-\lambda_D^2 \Delta V = 2n_0 - C \quad \text{in } \Omega \times (0, T), \quad (5)$$

where  $C = C(x)$  is the doping profile and  $\lambda_D > 0$  is the (constant) scaled Debye length. For physical reasons, we require that  $\mu$  satisfies:

$$\mu \in C^1([0, \infty)), \quad \mu'(s) \leq 0, \quad s\mu(s) \leq v^{sat}, \quad |s\mu(s) - \bar{s}\mu(\bar{s})| \leq L|s - \bar{s}| \quad \forall s, \bar{s} \geq 0, \quad (6)$$

for some constants  $v^{sat} > 0$  (called *saturation velocity*),  $L > 0$ . Eqs. (4), (6) imply that the scaled drift velocity  $v$  of the electrons, which is parallel to the electric field  $-\nabla V$ , remains bounded even when  $|\nabla V|$  is large. This fact is in agreement with the experimental evidence that the charge carriers velocity in a material cannot be higher than the saturation velocity due to collisions with the acoustic phonons [19]. We impose the following Dirichlet boundary conditions on  $n_0$ ,  $\vec{n}$ ,  $V$ , and initial conditions on  $n_0$ ,  $\vec{n}$ :

$$n_0 = n_0^D, \quad \vec{n} = \vec{n}^D, \quad V = V^D, \quad \text{on } \partial\Omega \times (0, T), \quad (7)$$

$$n_0(\cdot, 0) = n_0^I, \quad \vec{n}(\cdot, 0) = \vec{n}^I \quad \text{on } \Omega. \quad (8)$$

The magnetization vector  $\vec{m}$  has constant modulus and satisfies the Landau-Lifshitz equation [1, 6]:

$$\partial_t \vec{m} = \vec{m} \wedge \Delta \vec{m} - \vec{m} \wedge (\vec{m} \wedge \Delta \vec{m}), \quad |\vec{m}| \equiv 1 \quad \text{in } \Omega \times (0, T). \quad (9)$$

We impose the following homogeneous Neumann boundary conditions and initial conditions on  $\vec{m}$  [2]:

$$\partial_\nu \vec{m} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad \vec{m}(\cdot, 0) = \vec{m}^I \quad \text{on } \Omega. \quad (10)$$

The diffusion matrix  $A = (a_{ij})_{i,j=0,1,2,3}$  of the system,

$$A = D\eta^{-2} \begin{pmatrix} 1 & -p\vec{m}^T \\ -p\vec{m} & \eta I + (1-\eta)\vec{m} \otimes \vec{m} \end{pmatrix}, \quad (11)$$

which depends on  $(x, t)$  through  $\vec{m}$  and  $p$ , is symmetric and positive definite. In fact, its spectral decomposition is:

$$A = \frac{D}{1+p} \Pi^+ + \frac{D}{1-p} \Pi^- + \frac{D}{\eta} \Pi^\perp, \quad (12)$$

where:

$$\Pi^\pm = \frac{1}{2} \begin{pmatrix} 1 & \pm \vec{m}^T \\ \pm \vec{m} & \vec{m} \otimes \vec{m} \end{pmatrix}, \quad \Pi^\perp = \begin{pmatrix} 0 & 0 \\ 0 & I - \vec{m} \otimes \vec{m} \end{pmatrix}, \quad (13)$$

are the projections operators in the eigenspaces related to the eigenvalues of  $A$ . In particular, the eigenvalues of  $A$ , namely  $D/(1+p)$ ,  $D/(1-p)$ ,  $D/\eta$ , do not depend on  $\vec{m}$  and are thus time-independent.

## 2. Theoretical considerations

The derivation of system (1)–(3) from a linear Boltzmann equation with matrix collision operator can be found in [17]. A similar model has been considered in [11] (Chapter 6.2): the system of parabolic PDEs for particle density and spin vector given by eq. (6.2.2) is formally identical to eqs. (1)–(3) with  $\eta = 1$ . Moreover, in the model presented in [11], the magnetization is given by a Landau-Lifshitz equation with an additional term, proportional to the cross product between the spin and the magnetization, which provides a weak coupling between the Landau-Lifshitz equation and the drift-diffusion equations. Other drift-diffusion models for spin-polarized electron transport have been derived in [8] from a spinor Boltzmann equation with spin-flip and non spin-flip collision operators under the hypothesis that the former is a small perturbation of the latter, and in [18] from a Wigner equation with semiclassical matrix collision operator under the assumption that spin-orbit coupling is small in comparison with the electron kinetic energy. Both models do not contain any cross diffusion term and consist in a set of decoupled linear parabolic equations for the particle density and the spin density vector.

Eqs. (1)–(3) have been analytically studied in [12] in the case of given constant magnetization  $\vec{m}$  and mobility  $\mu \equiv 1$ . In such a case the cross diffusion terms in eqs. (1)–(3) can be removed, thus significantly weakening the coupling between equations, by considering the variables:

$$n_+ = n_0 + \vec{m} \cdot \vec{n}, \quad n_- = n_0 - \vec{m} \cdot \vec{n}, \quad \vec{n}_\perp = \vec{n} - (\vec{n} \cdot \vec{m})\vec{m}. \quad (14)$$

The quantities  $n_+$  and  $n_-$  are referred to in [12] as spin-up and spin-down densities, respectively. In particular, the equations for  $n_+$ ,  $n_-$  and  $\vec{n}_\perp$  are decoupled:

$$\partial_t n_\pm - \partial_i \left( \frac{D}{1 \pm p} (\partial_i n_\pm + n_\pm \partial_i V) \right) \pm \frac{n_+ - n_-}{2\tau} = 0, \quad (15)$$

$$\partial_t \vec{n}_\perp - \partial_i \left( \frac{D}{\eta} (\partial_i \vec{n}_\perp + \vec{n}_\perp \partial_i V) \right) - 2\gamma \vec{n}_\perp \wedge \vec{m} + \frac{\vec{n}_\perp}{\tau} = 0. \quad (16)$$

Thanks to the simple structure of eqs. (15), (16), the existence of bounded solutions to eqs. (1)–(5), with positive densities  $n_\pm$ , is shown in [12] under the assumption that the diffusion coefficient  $D$  and the polarization  $p$  are constant. The same result is proved for system (1)–(4) when  $D$  and  $p$  are bounded functions of  $x$ , and  $V = V(x)$  is a given function with bounded gradient.

The mathematical analysis of eqs. (1)–(10) is challenging because of the presence of cross diffusion terms with nonconstant and nonsmooth coefficients, unlike the problem considered in [12]. Since in this paper the magnetization  $\vec{m} = \vec{m}(x, t)$  is given by eq. (9), we cannot rely on the above argument to prove existence, boundedness and positivity of the solutions to eqs. (1)–(8). In fact, for a nonconstant magnetization the right-hand sides of eqs. (15), (16) are nonzero and the equations are coupled. As a consequence, results of positivity or boundedness for  $n_\pm$  cannot be achieved with usual methods (like e.g. Stampacchia truncation technique [20]). Moreover, since in the case at hand eqs. (15), (16) would depend also on the derivatives of  $\vec{m}$ , they could not be exploited to prove existence of solutions for system (1)–(8) unless strong regularity assumptions on  $\vec{m}$  are assumed. Also a constant mobility, which has been treated in [12], creates huge problems to the mathematical study of eqs. (1)–(8) in presence of nonconstant cross diffusion terms, because the particular (quadratic) structure of the drift terms  $n_j \nabla V$  ( $0 \leq j \leq 3$ ), with  $V$  given by the Poisson equation (5), cannot be exploited with standard techniques to derive entropy estimates (see e.g. [10, 12, 21]). The (physically justified) velocity saturation assumption allows us to overcome this difficulty, by providing an upper bound for the drift velocity modulus. We defer the study of eqs. (1)–(5) with a constant mobility to a future work.

## 3. Main results

Now we will state our main results and explain the ideas of the proofs. Let  $\partial\Omega \in C^1$ . To fix a convenient notation let us define:

$$n = (n_0, n_1, n_2, n_3), \quad n^D = (n_0^D, n_1^D, n_2^D, n_3^D), \quad n^I = (n_0^I, n_1^I, n_2^I, n_3^I),$$

$$B = (b_{ij})_{i,j=0,1,2,3}, \quad b_{ij} = \begin{cases} 2\gamma\varepsilon_{ijk}m_k - \tau^{-1}\delta_{ij} & 1 \leq i \leq 3, 1 \leq j \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

We can then rewrite eqs. (1)–(5), (7), (8) in the following synthetic form:

$$\partial_t n_i = \operatorname{div}(a_{ij}(\nabla n_j - n_j v)) + b_{ij}n_j \quad (0 \leq i \leq 3) \quad \text{in } \Omega \times (0, T), \quad (17)$$

$$v = -\mu(|\nabla V|)\nabla V \quad \text{in } \Omega \times (0, T), \quad (18)$$

$$-\lambda_D^2 \Delta V = 2n_0 - C \quad \text{in } \Omega \times (0, T), \quad (19)$$

$$n = n^D, \quad V = V^D \quad \text{on } \partial\Omega \times (0, T), \quad (20)$$

$$n(\cdot, 0) = n^I \quad \text{on } \Omega. \quad (21)$$

For the analysis of eqs. (17)–(21) we exploited the following result concerning the solutions of the Landau-Lifshitz equation, which is the content of Theorem 1.5 in [2] and Theorem 1.4 in [5].

**Proposition 1.** *For all  $\vec{m}^I \in H^1(\Omega)$  with  $|\vec{m}^I| \equiv 1$  in  $\Omega$ , a solution  $\vec{m}$  to eqs. (9), (10) exists such that:*

$$\vec{m} \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \quad \forall T > 0. \quad (22)$$

Moreover if  $\Omega \subset \mathbb{R}^2$ , then a number  $r > 0$  exists such that, for all  $\vec{m}^I \in H^2(\Omega)$  with  $|\vec{m}^I| \equiv 1$  in  $\Omega$ ,  $\partial_\nu \vec{m}^I = 0$  on  $\partial\Omega$ ,  $\|\nabla \vec{m}^I\|_{H^1(\Omega)} \leq r$ , a unique solution to eqs. (9), (10) exists satisfying:

$$\vec{m} \in C^0([0, T]; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \quad \forall T > 0. \quad (23)$$

The first result we present is the following existence and uniqueness theorem.

**Theorem 1 (Existence and uniqueness).** *Let  $\lambda_D > 0$ ,  $\gamma > 0$ ,  $T > 0$ . Moreover let us assume that  $\vec{m}$  satisfies eq. (22) and*

$$n_j^I \in L^2(\Omega), \quad n_j^D, V^D \in H^1(\Omega), \quad C \in L^\infty(\Omega) \quad (0 \leq j \leq 3), \quad (24)$$

$$D \in L^\infty(\Omega), \quad \inf_\Omega D > 0, \quad \sup_\Omega |p| < 1. \quad (25)$$

Then problem (17)–(21) has a unique solution  $(n, V)$  satisfying:

$$n_j \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \quad (0 \leq j \leq 3), \quad (26)$$

$$V \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)). \quad (27)$$

The idea for the proof is to discretize eqs. (17)–(21) in time with the implicit Euler method with time step  $h > 0$ . The problem we obtain is (here we neglect the boundary conditions (20) for the sake of simplicity):

$$\begin{aligned} n_i^{(k)} - h \operatorname{div}(a_{ij}^{(k)} \nabla n_j^{(k)}) &= n_i^{(k-1)} - h \operatorname{div}(a_{ij}^{(k)} n_j^{(k)} v^{(k)}) + h b_{ij}^{(k)} n_j^{(k)} \quad (0 \leq i \leq 3) \quad \text{in } \Omega, \\ v^{(k)} &= -\mu(|\nabla V^{(k)}|) \nabla V^{(k)} \quad \text{in } \Omega, \quad -\lambda_D^2 \Delta V^{(k)} = 2n_0^{(k)} - C \quad \text{in } \Omega, \end{aligned}$$

which can be solved by a standard fixed point argument. To take the limit  $h \rightarrow 0$ , thus solving the original problem (17)–(21), we derive a discrete entropy inequality for the following discrete quadratic entropy functional:

$$S[n^{(k)}] = \frac{1}{2} \int_\Omega |n^{(k)} - n^D|^2.$$

The inequality has the form:

$$h^{-1}(S[n^{(k)}] - S[n^{(k-1)}]) + c_0 \int_\Omega |\nabla n^{(k)}|^2 \leq c(S[n^{(k)}] + 1),$$

and yields gradient estimates for the solution, which allow us to take the limit for vanishing step size and so to obtain a solution of eqs. (17)–(21). The uniqueness of solutions is achieved by deriving an entropy inequality for the relative entropy

$$S[n, \bar{n}] = \frac{1}{2} \int_{\Omega} |n - \bar{n}|^2,$$

where  $n, \bar{n}$  are two solutions with the same initial and boundary data. Such inequality reads:

$$\frac{d}{dt} S[n, \bar{n}] \leq c(1 + \|\bar{n}\|_{H^1(\Omega)}^2) S[n, \bar{n}] \quad t > 0.$$

Since  $S[n, \bar{n}](t = 0) = 0$ ,  $S[n, \bar{n}] \in L^\infty(0, T)$  and  $\|\bar{n}\|_{H^1(\Omega)}^2 \in L^1(0, T)$ , the above estimate allows to deduce  $n = \bar{n}$  once that the Gronwall lemma is applied.

The second result we present concerns the behaviour for  $t \rightarrow \infty$  of the solution to eqs. (17)–(21). The steady state solution  $(n^{eq}, V^{eq})$  for the system is defined by:

$$\nabla n_0^{eq} + n_0^{eq} \mu(|\nabla V^{eq}|) \nabla V^{eq} = 0 \quad \text{in } \Omega, \quad (28)$$

$$\lambda_D^2 \Delta V^{eq} + 2n_0^{eq} - C = 0 \quad \text{in } \Omega, \quad (29)$$

$$n_0^{eq} = n_0^D, \quad V^{eq} = V^D \quad \text{on } \partial\Omega, \quad (30)$$

$$\vec{n}^{eq} = 0 \quad \text{in } \Omega. \quad (31)$$

We point out that, if  $u^{eq} \equiv \log n_0^{eq}$ , eqs. (28)–(30) can be (formally) rewritten as:

$$\Delta u^{eq} + \operatorname{div}(\mu(|\nabla V^{eq}|) \nabla V^{eq}) = 0 \quad \text{in } \Omega, \quad (32)$$

$$\lambda_D^2 \Delta V^{eq} + 2 \exp(u^{eq}) - C = 0 \quad \text{in } \Omega, \quad (33)$$

$$u^{eq} = u^D \equiv \log n_0^D, \quad V^{eq} = V^D \quad \text{on } \partial\Omega, \quad (34)$$

$$\operatorname{curl}(\mu(|\nabla V^{eq}|) \nabla V^{eq}) = 0 \quad \text{in } \Omega. \quad (35)$$

While eqs. (32)–(34) can be solved with standard techniques for nonlinear elliptic equations, eq. (35) constitutes a nonlinear constraint for the solution  $(n^{eq}, V^{eq})$  of eqs. (32)–(34), thus making the analytical study of the steady state problem tricky. For this reason, in this paper we just assume that there is a steady state  $(n^{eq}, V^{eq})$  and prove the convergence of the solution  $(n, V)$  of eqs. (17)–(21) to  $(n^{eq}, V^{eq})$  as  $t \rightarrow \infty$ . We defer the study of eqs. (32)–(35) to a future work.

**Theorem 2 (Convergence to the steady state).** *Let  $(n^{eq}, V^{eq}) \in H^1(\Omega)^4 \times H^1(\Omega)$  satisfy eqs. (28)–(31). A constant  $K > 0$  depending on  $D, p, \Omega$  exists such that, if:*

$$(v^{sat})^2 + L^2 \lambda_D^{-4} \|n^{eq}\|_{L^\infty(\Omega)}^2 < K,$$

where  $L$  is the constant in eq. (6), then:

$$\|n(t) - n^{eq}\|_{L^2(\Omega)} \leq k_1 e^{-k_2 t} \quad t > 0, \quad (36)$$

for suitable constants  $k_1, k_2 > 0$ .

The idea for the proof is to derive an entropy inequality for the relative entropy:

$$S[n, n^{eq}] = \frac{1}{2} \int_{\Omega} |n - n^{eq}|^2,$$

which reads:

$$\frac{d}{dt} \int_{\Omega} S[n, n^{eq}] + c_0 \int_{\Omega} |\nabla(n - n^{eq})|^2 \leq c((v^{sat})^2 + L^2 \lambda_D^{-4} \|n^{eq}\|_{L^\infty(\Omega)}^2) S[n, n^{eq}]. \quad (37)$$

Poincaré Lemma implies that the entropy  $S[n, n^{eq}]$  can be controlled by the entropy dissipation  $\int_{\Omega} |\nabla(n - n^{eq})|^2$ . So, under a suitable smallness assumption on the constant  $(v^{sat})^2 + L^2 \lambda_D^{-4} \|n^{eq}\|_{L^\infty(\Omega)}^2$ , the entropy inequality (37) and Gronwall lemma imply eq. (36).

To prove the last Theorem, concerning the regularity of the solutions to eqs. (17)–(21), we need the following result (an immediate corollary of Theorem 1.1 in [3]), which is a generalization of the well-known Aubin lemma.

**Proposition 2.** *Let  $E, E_0, E_1$  Banach spaces such that the embedding  $E_1 \hookrightarrow E_0$  is compact and the embeddings  $E_1 \hookrightarrow E, E \hookrightarrow E_0$  are continuous. Moreover let us assume that constants  $c > 0, \theta \in (0, 1)$  exist such that:*

$$\|u\|_E \leq c \|u\|_{E_0}^{1-\theta} \|u\|_{E_1}^\theta \quad \forall u \in E_1. \quad (38)$$

*Then, for all  $p \in [1, \infty]$ , the embedding  $L^p(0, T; E_1) \cap W^{1,p}(0, T; E_0) \hookrightarrow C([0, T]; E)$  is compact.*

**Theorem 3 (Improved regularity).** *Let  $\Omega \subset \mathbb{R}^2, \lambda_D > 0, \gamma > 0, T > 0$ . Moreover let us assume that  $\vec{m}$  satisfies eq. (23) and*

$$n_j^I \in H^1(\Omega), \quad n_j^D, V^D \in H^2(\Omega), \quad C \in L^\infty(\Omega) \quad (0 \leq j \leq 3), \quad (39)$$

$$D, p \in W^{1,\infty}(\Omega), \quad \inf_{\Omega} D > 0, \quad \sup_{\Omega} |p| < 1. \quad (40)$$

*Then the unique solution  $(n, V)$  to problem (17)–(21) satisfies:*

$$n_j \in L^q(0, T; W^{2,q}(\Omega)) \cap W^{1,q}(0, T; L^q(\Omega)) \cap C([0, T]; W^{1,q}(\Omega)) \quad (0 \leq j \leq 3), \quad (41)$$

$$V \in L^q(0, T; W^{3,q}(\Omega)) \cap W^{1,q}(0, T; W^{1,q}(\Omega)) \cap C([0, T]; W^{2,q}(\Omega)), \quad (42)$$

*for all  $q \in [1, 2)$ .*

The idea of the proof is that the regularity assumptions (23) on  $\vec{m}$ , stronger than the hypothesis (22) made for the existence analysis, allow us to derive and exploit a set of equations for the variables  $n_+, n_-, \vec{n}_\perp$  defined in eq. (14). Such equations are a generalization of eqs. (9), (10), (13) in [12] and have the form:

$$\partial_t n_\pm - \partial_i \left( \frac{D}{1 \pm p} (\partial_i n_\pm - v_i n_\pm) \right) = f_\pm, \quad \partial_t \vec{n}_\perp - \partial_i \left( \frac{D}{\eta} (\partial_i \vec{n}_\perp - v_i \vec{n}_\perp) \right) + \frac{\vec{n}_\perp}{\tau} = \vec{f}_\perp, \quad (43)$$

with  $f_\pm, \vec{f}_\perp$  suitable quantities depending from  $n, \nabla n, \vec{m}, \partial_t \vec{m}, \nabla \vec{m}, \Delta \vec{m}, v$ . The big advantage in eqs. (43) consists in the lack of cross diffusion terms, which makes possible to exploit the above equations in order to derive improved regularity results for the solutions of pb. (17)–(21). The proof of the Theorem consists in three parts. In the first part eq. (43) is derived. In the second part eqs. (41), (42) are shown for  $p = 4/3$ . Hypothesis  $\Omega \subset \mathbb{R}^2$  allows to employ the following Gagliardo-Nirenberg inequality [4]:

$$\|u\|_{L^4} \leq c \|u\|_{L^2}^{1/2} \|u\|_{H^1}^{1/2} \quad u \in H^1(\Omega), \quad (44)$$

which, together with the already known regularity results (26) on  $n$ , implies that  $n \in L^4(\Omega \times (0, T))$ . From this fact and the assumptions (23) on  $\vec{m}$ , we are able to deduce that the source terms  $f_\pm, \vec{f}_\perp$  in eq. (43) belong to  $L^{4/3}(\Omega \times (0, T))$ . This property, thanks to suitable regularity results for parabolic problems [4, 9, 15], yields eqs. (41), (42) for  $p = 4/3$ . In the third part of the proof, eqs. (41), (42) are shown to hold for an increasing sequence of exponents  $(p_k)_{k \in \mathbb{N}} \subset (1, 2)$ . The partial result obtained in the previous part of the proof provides the first step in the derivation of such a sequence ( $p_0 = 4/3$ ), which is built by iteratively applying Proposition 2 with  $E_0 = L^{p_k}(\Omega), E = W_0^{1,p_k}(\Omega), E_1 = W^{2,p_k} \cap W_0^{1,p_k}(\Omega)$ . In particular, assumption (38) is a consequence of the following Gagliardo-Nirenberg inequality [4]:

$$\|\nabla u\|_{L^p} \leq c \|u\|_{L^p}^{1/2} \|u\|_{W^{2,p}}^{1/2} \quad u \in W^{2,p}(\Omega), \quad (45)$$

valid for  $p \in [1, \infty]$ . The fact that  $p_k \rightarrow 2$  as  $k \rightarrow \infty$  implies that eqs. (41), (42) hold for all  $p < 2$ .

The paper is organized as follows. We prove Theorem 1 in Section 4. Theorem 2 is proved in Section 5. Finally, Section 6 is devoted to the proof of Theorem 3.

#### 4. Existence and uniqueness of weak solutions

In this section we prove Theorem 1. The proof is divided into four steps.

*Step 1: time semi-discretization.* Let  $h > 0$  be the time step,  $t_k = kh$ ,  $n^{(k)} = n(\cdot, t_k)$ ,  $V^{(k)} = V(\cdot, t_k)$  for  $0 \leq k \leq T/h$ . We consider the following semi-discretized problem:

$$n_i^{(k)} - h \operatorname{div}(a_{ij}^{(k)} \nabla n_j^{(k)}) = n_i^{(k-1)} - h \operatorname{div}(a_{ij}^{(k)} n_j^{(k)} v^{(k)}) + h b_{ij}^{(k)} n_j^{(k)} \quad (0 \leq i \leq 3) \quad \text{in } \Omega, \quad (46)$$

$$v^{(k)} = -\mu(|\nabla V^{(k)}|) \nabla V^{(k)} \quad \text{in } \Omega, \quad (47)$$

$$-\lambda_D^2 \Delta V^{(k)} = 2n_0^{(k)} - C \quad \text{in } \Omega, \quad (48)$$

$$n^{(k)} = n^D, \quad V^{(k)} = V^D \quad \text{on } \partial\Omega, \quad (49)$$

for  $k \geq 1$ .

*Step 2: fixed point.* We solve eqs. (46)–(49) by applying Leray-Schauder's fixed point theorem [22]. Let us define the operator

$$F : \begin{array}{ccc} L^2(\Omega) \times [0, 1] & \rightarrow & L^2(\Omega) \\ (n, \sigma) & \mapsto & u \end{array}$$

where  $u \in H^1(\Omega)$  satisfies:

$$u_i - h \operatorname{div}(a_{ij}^{(k)} \nabla u_j) = n_i^{(k-1)} - \sigma h \operatorname{div}(a_{ij}^{(k)} n_j v) + \sigma h b_{ij}^{(k)} n_j \quad (0 \leq i \leq 3) \quad \text{in } \Omega, \quad (50)$$

$$v = -\mu(|\nabla V|) \nabla V \quad \text{in } \Omega, \quad (51)$$

$$-\lambda_D^2 \Delta V = 2n_0 - C \quad \text{in } \Omega, \quad (52)$$

$$u = n^D, \quad V = V^D \quad \text{on } \partial\Omega. \quad (53)$$

The existence of a unique solution  $u \in H^1(\Omega)$  to eqs. (50)–(53) is an easy application of Lax-Milgram lemma [4]. The operator  $F$  is compact due to the compact Sobolev embedding  $H^1(\Omega) \hookrightarrow L^2(\Omega)$ . The continuity of  $F$  can be proved with a standard argument (see e.g. [13]). Moreover  $F(\cdot, 0)$  is constant. Finally, let  $n \in L^2(\Omega)$ ,  $\sigma \in [0, 1]$  such that  $F(n, \sigma) = n$ . Let us consider the quadratic entropy functional:

$$S[n] = \frac{1}{2} \int_{\Omega} |n - n^D|^2. \quad (54)$$

From eq. (50) and the fact that  $A$  is symmetric and positive definite it follows:

$$\begin{aligned} h^{-1}(S[n] - S[n^{(k-1)}]) &\leq \int_{\Omega} (n_j - n_j^D) h^{-1}(n_j - n_j^{(k-1)}) \\ &= - \int_{\Omega} (a_{ij}^{(k)} \nabla n_j - \sigma a_{ij}^{(k)} n_j v) \cdot \nabla (n_i - n_i^D) + \int_{\Omega} \sigma b_{ij}^{(k)} n_j (n_i - n_i^D) \\ &\leq -\frac{1}{2} \int_{\Omega} a_{ij}^{(k)} \nabla n_i \cdot \nabla n_j + \frac{1}{2} \int_{\Omega} a_{ij}^{(k)} \nabla n_i^D \cdot \nabla n_j^D + c \int_{\Omega} |n| (|\nabla(n - n^D)| + |n - n^D|), \end{aligned}$$

so by exploiting the positivity and boundedness of  $A$  and the assumptions (24) on the data and by applying Young inequality we obtain:

$$h^{-1}(S[n] - S[n^{(k-1)}]) + c_0 \int_{\Omega} |\nabla n|^2 \leq c(S[n] + 1), \quad (55)$$

for some positive constants  $c_0, c$ , independent on  $\sigma, h$ .

Eq. (55) yields  $\sigma$ -uniform bounds for  $n, \nabla n$  in  $L^2(\Omega)$ . So from Leray-Schauder's theorem (see e.g. [22]) we obtain the existence of a fixed point  $n^{(k)} \in H^1(\Omega)$  for  $F(\cdot, 1)$ , which means, a solution of (46)–(49).

*Step 3: limit  $h \rightarrow 0$ .* Eq. (55) holds with  $n = n^{(k)}$ :

$$h^{-1}(S[n^{(k)}] - S[n^{(k-1)}]) + c_0 \int_{\Omega} |\nabla n^{(k)}|^2 \leq c(S[n^{(k)}] + 1). \quad (56)$$

Let us define the piecewise constant functions:  $n^h(\cdot, t) = n^{(k)}$ ,  $V^h(\cdot, t) = V^{(k)}$  for  $t \in ((k-1)h, kh]$ . From eq. (56) we deduce:

$$S[n^h(t)] + c_0 \int_0^t \int_{\Omega} |\nabla n^h|^2 \leq c_1 \int_0^t S[n^h(s)] ds + c(T), \quad t \in [0, T]. \quad (57)$$

By applying Gronwall's lemma, from eq. (57) we get:

$$\|n^h\|_{L^2(0,T;H^1(\Omega))} + \|n^h\|_{L^\infty(0,T;L^2(\Omega))} \leq c. \quad (58)$$

If we define the discrete time derivative  $D_h w$  of an arbitrary function  $u : \Omega \times (0, T) \rightarrow \mathbb{R}$  as:

$$D_h w(x, t) = (w(x, t) - w(x, t - h))/h \quad (x, t) \in \Omega \times (h, T),$$

from eqs. (46), (58) it follows easily:

$$\|D_h n^h\|_{L^2(h,T;H^{-1}(\Omega))} \leq c. \quad (59)$$

By exploiting eqs. (58), (59) and the compact embedding  $H^1(\Omega) \hookrightarrow L^p(\Omega)$  for all  $p \in [1, 6)$ , from Theorem 1 in [7] we argue that, up to subsequences,  $n^h \rightarrow n$  in  $L^2(0, T; L^p(\Omega))$  as  $h \rightarrow 0$ , for all  $p \in [1, 6)$ . Since eq. (48) holds, this implies  $V^h \rightarrow V$  in  $L^2(0, T; W^{2,p}(\Omega))$  as  $h \rightarrow 0$ ; in particular,  $\nabla V^h \rightarrow \nabla V$  a.e. in  $\Omega \times (0, T)$  as  $h \rightarrow 0$ , so from the dominated convergence theorem  $v^k \rightarrow v = -\mu(|\nabla V|)\nabla V$  in  $L^q(\Omega \times (0, T))$  for all  $q \in [1, \infty)$ . From the pointwise convergence of  $n^h$  and Fatou's lemma we deduce that  $n \in L^\infty(0, T; L^2(\Omega))$ . By exploiting these properties it is straightforward to take the limit  $h \rightarrow 0$  in eqs. (46)–(49), proving that  $(n, V)$  satisfies eqs. (17)–(21), (26), (27).

*Step 4: uniqueness.* Let  $(n, V)$ ,  $(\bar{n}, \bar{V})$  solutions to eqs. (17)–(21) satisfying eqs. (26), (27). Moreover let  $v = -\mu(|\nabla V|)\nabla V$ ,  $\bar{v} = -\mu(|\nabla \bar{V}|)\nabla \bar{V}$ . Then:

$$\begin{aligned} \partial_t(n_i - \bar{n}_i) &= \operatorname{div}(a_{ij}(\nabla(n_j - \bar{n}_j) - (n_j - \bar{n}_j)v - \bar{n}_j(v - \bar{v}))) \\ &\quad + b_{ij}(n_j - \bar{n}_j) \quad (0 \leq i \leq 3) \quad \text{in } \Omega \times (0, T), \end{aligned} \quad (60)$$

$$-\lambda_D^2 \Delta(V - \bar{V}) = n_0 - \bar{n}_0 \quad \text{in } \Omega \times (0, T), \quad (61)$$

$$n - \bar{n} = 0, \quad V - \bar{V} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (62)$$

$$n(\cdot, 0) - \bar{n}(\cdot, 0) = 0 \quad \text{on } \Omega. \quad (63)$$

Because of eq. (6) it holds that  $|v - \bar{v}| \leq L|\nabla(V - \bar{V})|$ . So, if we use  $n - \bar{n}$  as a test function in the weak formulation of eq. (60) we find:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \frac{1}{2} |n - \bar{n}|^2 + c_0 \int_{\Omega} |\nabla(n - \bar{n})|^2 & \\ \leq c \int_{\Omega} |n - \bar{n}| |\nabla(n - \bar{n})| + c \int_{\Omega} |\bar{n}| |\nabla(V - \bar{V})| |\nabla(n - \bar{n})| & \\ \leq \varepsilon \int_{\Omega} |\nabla(n - \bar{n})|^2 + \frac{c}{\varepsilon} \int_{\Omega} (|n - \bar{n}|^2 + |\bar{n}|^2 |\nabla(V - \bar{V})|^2) & \\ \leq \varepsilon \int_{\Omega} |\nabla(n - \bar{n})|^2 + \frac{c}{\varepsilon} (\|n - \bar{n}\|_{L^2(\Omega)}^2 + \|\bar{n}\|_{L^4(\Omega)}^2 \|\nabla(V - \bar{V})\|_{L^4(\Omega)}^2). & \end{aligned} \quad (64)$$

From the Sobolev embedding  $H^1(\Omega) \subset L^4(\Omega)$  and eq. (61) it follows:

$$\|\bar{n}\|_{L^4(\Omega)}^2 \|\nabla(V - \bar{V})\|_{L^4(\Omega)}^2 \leq c \|\bar{n}\|_{H^1(\Omega)}^2 \|V - \bar{V}\|_{H^2(\Omega)}^2 \leq c \|\bar{n}\|_{H^1(\Omega)}^2 \|n - \bar{n}\|_{L^2(\Omega)}^2,$$

so eq. (64) can be rewritten as:

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} |n - \bar{n}|^2 + c_0 \int_{\Omega} |\nabla(n - \bar{n})|^2 \leq \varepsilon \int_{\Omega} |\nabla(n - \bar{n})|^2 + \frac{c}{\varepsilon} (1 + \|\bar{n}\|_{H^1(\Omega)}^2) \|n - \bar{n}\|_{L^2(\Omega)}^2.$$



If we choose  $\varepsilon > 0$  small enough we obtain:

$$\frac{d}{dt} \|n - \bar{n}\|_{L^2(\Omega)}^2 \leq c(1 + \|\bar{n}\|_{H^1(\Omega)}^2) \|n - \bar{n}\|_{L^2(\Omega)}^2 \quad t > 0. \quad (65)$$

Eq. (26) implies that  $\|\bar{n}\|_{H^1(\Omega)}^2 \in L^1(0, T)$ , thus from Gronwall's inequality we deduce that  $n = \bar{n}$  in  $\Omega \times (0, T)$ . From eqs. (61), (62) we also get  $V = \bar{V}$  in  $\Omega \times (0, T)$ . This finishes the proof.

## 5. Long-time behaviour of solutions

In this section we prove Theorem 2.

Let us employ  $n - n^{eq}$  as test function in the weak formulation of eqs. (17)–(21). We obtain:

$$\frac{d}{dt} \int_{\Omega} \frac{|n - n^{eq}|^2}{2} + \int_{\Omega} \partial_i(n - n^{eq}) \cdot A \partial_i n - \int_{\Omega} v_i \partial_i(n - n^{eq}) \cdot A n - \int_{\Omega} (n - n^{eq}) \cdot B n = 0,$$

which, by applying eq. (28), can be rewritten as:

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \frac{|n - n^{eq}|^2}{2} + \int_{\Omega} \partial_i(n - n^{eq}) \cdot A \partial_i(n - n^{eq}) \\ & - \int_{\Omega} \partial_i(n - n^{eq}) \cdot A((n - n^{eq})v_i + n^{eq}(v_i - v_i^{eq})) - \int_{\Omega} (n - n^{eq}) \cdot B n = 0. \end{aligned} \quad (66)$$

Since  $A$  is strictly positive definite we deduce:

$$\int_{\Omega} \partial_i(n - n^{eq}) \cdot A \partial_i(n - n^{eq}) \geq c_0 \int_{\Omega} |\nabla(n - n^{eq})|^2, \quad (67)$$

for some  $c_0 > 0$ . Moreover, since  $|v| \leq v^{sat}$ :

$$\int_{\Omega} \partial_i(n - n^{eq}) \cdot A(n - n^{eq})v_i \leq c\varepsilon \int_{\Omega} |\nabla(n - n^{eq})|^2 + \frac{c}{\varepsilon} (v^{sat})^2 \int_{\Omega} |n - n^{eq}|^2. \quad (68)$$

From eqs. (19), (20), (29), (30) it follows:

$$-\lambda_D^2 \Delta(V - V^{eq}) = n_0 - n_0^{eq} \quad \text{in } \Omega \times (0, T), \quad (69)$$

$$V - V^{eq} = 0 \quad \text{on } \partial\Omega \times (0, T). \quad (70)$$

Thus eq. (6) and standard estimates for linear elliptic equations applied to eqs. (69), (70) imply:

$$\begin{aligned} & \int_{\Omega} \partial_i(n - n^{eq}) \cdot A n^{eq}(v_i - v_i^{eq}) \\ & \leq c\varepsilon \int_{\Omega} |\nabla(n - n^{eq})|^2 + \frac{c}{\varepsilon} L^2 \|n^{eq}\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla(V - V^{eq})|^2 \\ & \leq c\varepsilon \int_{\Omega} |\nabla(n - n^{eq})|^2 + \frac{c}{\varepsilon} L^2 \lambda_D^{-4} \|n^{eq}\|_{L^\infty(\Omega)}^2 \int_{\Omega} |n - n^{eq}|^2. \end{aligned} \quad (71)$$

Finally, from the definitions of  $B$  and  $n^{eq}$  it follows immediately:

$$- \int_{\Omega} (n - n^{eq}) \cdot B n = \int_{\Omega} \frac{|\vec{n}|^2}{\tau} \geq 0. \quad (72)$$

By collecting eqs. (66)–(68), (71), (72) and choosing  $\varepsilon > 0$  small enough we find:

$$\frac{d}{dt} \int_{\Omega} |n - n^{eq}|^2 + c_0 \int_{\Omega} |\nabla(n - n^{eq})|^2 \leq c((v^{sat})^2 + L^2 \lambda_D^{-4} \|n^{eq}\|_{L^\infty(\Omega)}^2) \int_{\Omega} |n_0 - n_0^{eq}|^2. \quad (73)$$

It is straightforward to deduce that the constants  $c_0$ ,  $c$  in eq. (73) depend only on  $D$  and  $p$ . Poincaré's Lemma implies that  $\int_{\Omega} |\nabla(n - n^{eq})|^2 \geq c_1 \int_{\Omega} |n - n^{eq}|^2$ , for a suitable constant  $c_1 > 0$  depending only on  $\Omega$ . So, if  $(v^{sat})^2 + L^2 \lambda_D^{-4} \|n^{eq}\|_{L^\infty(\Omega)}^2 < K \equiv c_0 c_1 / c$  then eq. (36) follows from Gronwall's Lemma applied to eq. (73). This finishes the proof.

## 6. Improved regularity

In this section we prove Theorem 3. The proof is divided into three steps.

*Step 1: diagonalization of the system.* By exploiting eqs. (17)–(21), we derive a set of equations, without cross diffusion terms, for the variables  $n_+$ ,  $n_-$ ,  $\vec{n}_\perp$  defined in eq. (14). Let us first multiply eq. (2) times  $m_k$  and sum over  $k = 1, 2, 3$ . By exploiting the fact that  $\partial_i \vec{m} \cdot \vec{m} = 0$  for  $i = 1, 2$  (thanks to eq. (9)) we get:

$$\partial_t(\vec{n} \cdot \vec{m}) - \vec{n} \cdot \partial_t \vec{m} - \partial_i \left( \frac{D}{\eta^2} (-pJ_0^i + m_s J_s^i) \right) + \frac{D}{\eta} J_s^i \partial_i m_s + \frac{\vec{n} \cdot \vec{m}}{\tau} = 0. \quad (74)$$

Now, if we compute the sum and the difference between eqs. (1), (74), we obtain equations for  $\partial_t n_+$ ,  $\partial_t n_-$ , respectively:

$$\partial_t n_\pm \mp \vec{n} \cdot \partial_t \vec{m} - \partial_i \left( \frac{D}{1 \pm p} (J_0^i \pm m_s J_s^i) \right) \pm \frac{D}{\eta} J_s^i \partial_i m_s \pm \frac{\vec{n} \cdot \vec{m}}{\tau} = 0. \quad (75)$$

By exploiting the simple relations:

$$J_0^i \pm m_s J_s^i = \partial_i n_\pm - v_i n_\pm \mp \vec{n} \cdot \partial_i \vec{m}, \quad J_s^i \partial_i m_s = \partial_i \vec{m} \cdot \partial_i \vec{n} - v_i \vec{n} \cdot \partial_i \vec{m},$$

we can rewrite eq. (75) as:

$$\begin{aligned} \partial_t n_\pm - \partial_i \left( \frac{D}{1 \pm p} (\partial_i n_\pm - v_i n_\pm) \right) &= \pm \vec{n} \cdot \partial_t \vec{m} \mp \partial_i \left( \frac{D}{1 \pm p} \vec{n} \cdot \partial_i \vec{m} \right) \\ &\mp \frac{D}{\eta} (\partial_i \vec{m} \cdot \partial_i \vec{n} - v_i \vec{n} \cdot \partial_i \vec{m}) \mp \frac{\vec{n} \cdot \vec{m}}{\tau}. \end{aligned} \quad (76)$$

Now let us multiply eq. (74) times  $m_k$ :

$$\begin{aligned} m_k \partial_t (\vec{n} \cdot \vec{m}) - m_k \vec{n} \cdot \partial_t \vec{m} - \partial_i \left( \frac{D}{\eta^2} m_k (-pJ_0^i + m_s J_s^i) \right) \\ + \frac{D}{\eta^2} (-pJ_0^i + m_s J_s^i) \partial_i m_k + \frac{D}{\eta} J_s^i m_k \partial_i m_s + m_k \frac{\vec{n} \cdot \vec{m}}{\tau} = 0 \quad (k = 1, 2, 3). \end{aligned} \quad (77)$$

Since:

$$\partial_t (n_\perp)_k = \partial_t n_k - m_k \partial_t (\vec{n} \cdot \vec{m}) - (\vec{n} \cdot \vec{m}) \partial_t m_k,$$

by taking the difference between eqs. (2), (77) we get:

$$\begin{aligned} \partial_t (n_\perp)_k + (\vec{n} \cdot \vec{m}) \partial_t m_k + m_k \vec{n} \cdot \partial_t \vec{m} - \partial_i \left( \frac{D}{\eta} (\delta_{ks} - m_k m_s) J_s^i \right) - 2\gamma \varepsilon_{ijk} n_i m_j \\ + \frac{D}{\eta^2} (pJ_0^i - m_s J_s^i) \partial_i m_k - \frac{D}{\eta} J_s^i m_k \partial_i m_s + \frac{(n_\perp)_k}{\tau} = 0 \quad (k = 1, 2, 3). \end{aligned} \quad (78)$$

Elementary computations imply that:

$$\begin{aligned} (\delta_{ks} - m_k m_s) J_s^i &= \partial_i (n_\perp)_k - v_i (n_\perp)_k + n_s \partial_i (m_k m_s), \\ (\vec{n} \cdot \vec{m}) \partial_t m_k + m_k \vec{n} \cdot \partial_t \vec{m} &= n_s \partial_t (m_k m_s), \end{aligned}$$

so from eq. (78) we conclude:

$$\begin{aligned} \partial_t (n_\perp)_k - \partial_i \left( \frac{D}{\eta} (\partial_i (n_\perp)_k - v_i (n_\perp)_k) \right) + \frac{(n_\perp)_k}{\tau} &= \partial_i \left( \frac{D}{\eta} n_s \partial_i (m_k m_s) \right) \\ &- \frac{D}{\eta^2} (p\delta_{k\ell} (\partial_i n_0 - v_i n_0) - m_\alpha (\delta_{\alpha\beta} \delta_{k\ell} + \eta \delta_{\alpha k} \delta_{\beta\ell}) (\partial_i n_\beta - v_i n_\beta)) \partial_i m_\ell \\ &- n_s \partial_t (m_k m_s) + 2\gamma \varepsilon_{ijk} n_i m_j \quad (k = 1, 2, 3). \end{aligned} \quad (79)$$

*Step 2: the thesis holds for  $p = 4/3$ .* We look now for estimates of the right-hand sides of eqs. (76), (79). In order to do this, let us observe that, since  $\Omega \subset \mathbb{R}^2$ , we can apply the following Gagliardo-Nirenberg inequality [4], valid for  $1 \leq q \leq p < \infty$ :

$$\|n_s(t)\|_{L^p(\Omega)} \leq c \|n_s(t)\|_{L^q(\Omega)}^{q/p} \|n_s(t)\|_{H^1(\Omega)}^{1-q/p} \quad (0 \leq s \leq 3) \quad \text{a.e. } t \in (0, T), \quad (80)$$

where  $c > 0$  is a suitable constant, depending only on  $\Omega$ . Since eq. (26) holds, then eq. (80) with  $p = 4$ ,  $q = 2$  implies:

$$\|n_s(t)\|_{L^4(\Omega)}^4 \leq \tilde{c}_s \|n_s(t)\|_{H^1(\Omega)}^2 \quad (0 \leq s \leq 3) \quad \text{a.e. } t \in (0, T), \quad (81)$$

where the constant  $\tilde{c}_s > 0$  is proportional to  $\|n_s\|_{L^\infty(0,T;L^2(\Omega))}^2 < \infty$ . By integrating eq. (81) in time and exploiting eq. (26) we deduce that  $n_s \in L^4(\Omega \times (0, T))$ , for  $0 \leq s \leq 3$ . We are going to exploit this fact and eq. (23) to prove that the right-hand sides of eqs. (76), (79) belong to  $L^{4/3}(\Omega \times (0, T))$ . Each term appearing in the right-hand sides of eqs. (76), (79) can be bounded (in modulus) by the function:

$$f \equiv c|n|(|\partial_t \vec{m}| + |\Delta \vec{m}| + |\nabla \vec{m}|^2 + 1) + c|\nabla n||\nabla \vec{m}|, \quad (82)$$

with  $c > 0$  a suitable constant. From eq. (23) it follows:

$$|\partial_t \vec{m}| + |\Delta \vec{m}| + |\nabla \vec{m}|^2 + 1 \in L^\infty(0, T; L^2(\Omega)), \quad |\nabla \vec{m}| \in L^\infty(0, T; L^p(\Omega)) \quad \forall p < \infty,$$

which, together with eq. (26) and the fact that  $n_s \in L^4(\Omega \times (0, T))$  for  $0 \leq s \leq 3$ , implies  $f \in L^{4/3}(\Omega \times (0, T))$ , so also the right-hand sides of eqs. (76), (79) belong to  $L^{4/3}(\Omega \times (0, T))$ . From this fact and the hypothesis (39) on the data, by applying standard regularity results for parabolic equations (see e.g. [4, 9, 15]) it follows that:

$$n_\pm, (n_\pm)_k \in L^{4/3}(0, T; W^{2,4/3}(\Omega)) \cap W^{1,4/3}(0, T; L^{4/3}(\Omega)) \quad (k = 1, 2, 3). \quad (83)$$

By exploiting eqs. (23), (83) and the inverse relations  $n_0 = (n_+ + n_-)/2$ ,  $\vec{n} = (n_+ - n_-)\vec{m}/2 + \vec{n}_\perp$ , we get:

$$n_s, \partial_t n_s, \partial_{x_i} n_s, \partial_{x_i x_j}^2 n_s \in L^{4/3}(\Omega \times (0, T)) \quad 0 \leq s \leq 3, \quad 1 \leq i, j \leq 2, \quad (84)$$

namely eq. (41) for  $p = 4/3$ . Eq. (42) for  $p = 4/3$  is a consequence of eqs. (5), (84).

*Step 3: the thesis holds for  $p < 2$ .* Now we prove the following:

**Claim 1.** *An increasing sequence  $(p_k)_{k \in \mathbb{N}} \subset (1, 2)$  exists such that:*

$$n_s, \partial_t n_s, \partial_{x_i} n_s, \partial_{x_i x_j}^2 n_s \in L^{p_k}(\Omega \times (0, T)) \quad 0 \leq s \leq 3, \quad 1 \leq i, j \leq 2, \quad (85)$$

for all  $k \geq 0$ .

*Proof.* The proof is by induction on  $k$ . Let  $p_0 \equiv 4/3$ , so the thesis is true for  $k = 0$  as a consequence of Step 1. Let us assume eq. (85) is true for some  $k \geq 0$ . Let us define the following reflexive Banach spaces:

$$E_1 = W^{2,p_k} \cap W_0^{1,p_k}(\Omega), \quad E = W_0^{1,p_k}(\Omega), \quad E_0 = L^{p_k}(\Omega).$$

From standard Sobolev embedding theorems it follows that the embeddings  $E_1 \hookrightarrow E$ ,  $E \hookrightarrow E_0$  are compact. Moreover Poincaré inequality and eq. (45) with  $p = p^k$  imply:

$$\|u\|_E \leq c \|\nabla u\|_{L^{p_k}} \leq c \|u\|_{E_0}^{1/2} \|u\|_{E_1}^{1/2} \quad \forall u \in E_1. \quad (86)$$

With this notation, eq. (85) means that  $n_s - n_s^D \in L^{p_k}(0, T; E_1)$  and  $\partial_t(n_s - n_s^D) \in L^{p_k}(0, T; E_0)$  for  $0 \leq s \leq 3$ . Thus, by applying Prop. 2, we deduce that  $n_s - n_s^D \in C([0, T]; E)$ , which implies, again by means of standard Sobolev embedding theorems, that:

$$n_s \in C([0, T]; W^{1,p_k}(\Omega)) \hookrightarrow L^\infty(0, T; L^{p_k^*}(\Omega)) \quad (0 \leq s \leq 3), \quad p_k^* \equiv \frac{2p_k}{2-p_k}. \quad (87)$$

From eqs. (26), (80) with  $q = p_k^*$ ,  $p = p_k^* + 2$  and (87) it follows that  $n_s \in L^{p_k^*+2}(\Omega \times (0, T))$ . By arguing like in the derivation of eq. (84) in step 2 we deduce that:

$$n_s, \partial_t n_s, \partial_{x_i} n_s, \partial_{x_i x_j}^2 n_s \in L^{p_{k+1}}(\Omega \times (0, T)) \quad 0 \leq s \leq 3, \quad 1 \leq i, j \leq 2, \quad (88)$$

with  $p_{k+1} = \left(\frac{1}{2} + \frac{1}{p_k^*+2}\right)^{-1}$ , which means:

$$p_{k+1} = \frac{4}{4 - p_k} \quad \forall k \geq 0, \quad p_0 = \frac{4}{3}. \quad (89)$$

By exploiting eq. (89) it is easy to prove that  $1 < p_k < p_{k+1} < 2$  for all  $k \geq 0$ . Thus the claim has been proved. □

From eqs. (5), (85), (87) it follows also:

$$V \in L^{p_k}(0, T; W^{3, p_k}(\Omega)) \cap W^{1, p_k}(0, T; W^{1, p_k}(\Omega)) \cap C([0, T]; W^{2, p_k}(\Omega)) \quad (k \geq 0). \quad (90)$$

The monotonicity of  $p_k$  implies that  $p_k$  has a finite limit  $\tilde{p}$  which must satisfy  $\tilde{p} = 4/(4 - \tilde{p})$ , so  $\tilde{p} = 2$ . From this fact and eqs. (85), (87) and (90) we conclude that eqs. (41), (42) hold for all  $q \in [1, 2)$ . This finishes the proof of Theorem 3.

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## References

- [1] A. Aharoni, Introduction to the Theory of Ferromagnetism, Vol. 109, Oxford University Press, 2000.
- [2] F. Alounges, A. Soyeur, On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness, *Nonlinear Analysis: Theory, Methods and Applications* 18 (1992) 1071–1084.
- [3] H. Amann, Compact embeddings of vector valued Sobolev and Besov spaces, *Glasnik matematički* 35 (2000) 161–177.
- [4] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2010.
- [5] G. Carbou, P. Fabrie, Regular solutions for Landau-Lifshitz Equation in a bounded domain, *Differential and integral equations* 14 (2001) 213–229.
- [6] S. Chikazumi, *Physics of Ferromagnetism*, second ed., Oxford University Press, 2009.
- [7] M. Dreher, A. Jüngel, Compact families of piecewise constant solutions in  $L^p(0, T; B)$ , *Nonlinear Analysis: Theory, Methods and Applications* 75 (2012), 3072–3077.
- [8] R. El Hajj, Diffusion models for spin transport derived from the spinor Boltzmann equation, *Comm. Math. Sci.*, in press (2013).
- [9] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, 1964.
- [10] H. Gajewski, K. Gröger, On the basic equations for carrier transport in semiconductors, *J. Math. Anal. Appl.* 113 (1986), 12–35.

- [11] C. Jourdana, Mathematical modeling and numerical simulation of innovative electronic nanostructures, Diss. Université de Toulouse, Université Toulouse III-Paul Sabatier, 2011.
- [12] A. Jüngel, C. Negulescu, P. Shpartko, Bounded weak solutions to a matrix drift-diffusion model for spin-coherent electron transport in semiconductors, Preprint (2013) arXiv:1312.2461.
- [13] A. Jüngel, On the existence and uniqueness of transient solutions of a degenerate nonlinear drift-diffusion model for semiconductors, *Mathematical Models and Methods in Applied Sciences* 4 (1994), 677-703.
- [14] A. Jüngel, *Transport Equations for Semiconductors*, Lect. Notes Phys. 773, Springer, Berlin, 2009.
- [15] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'Ceva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, 1968.
- [16] M. P. Marder, *Condensed matter physics*, John Wiley and Sons, 2010.
- [17] S. Possanner, C. Negulescu, Diffusion limit of a generalized matrix Boltzmann equation for spin-polarized transport, *Kinetic Related Models* 4 (2011), 1159-1191.
- [18] S. Saikin, A drift-diffusion model for spin-polarized transport in a two-dimensional non-degenerate electron gas controlled by spin-orbit interaction, *Journal of Physics: Condensed Matter* 16 (2004), 5071.
- [19] P. Y. Yu, M. Cardona, *Fundamentals of Semiconductors: Physics and Materials Properties*, Springer, New York, 2005.
- [20] N. Zamponi, A. Jüngel, Global existence analysis for degenerate energy-transport models for semiconductors, Preprint (2013) arXiv:1310.3377.
- [21] N. Zamponi, A. Jüngel, Two spinorial drift-diffusion models for quantum electron transport in graphene, *Commun. Math. Sci.* 11 (2013), 927-950.
- [22] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Vol. II/A, Springer, New York, 1990.